Diffuse orthogonally additive operators in vector lattices

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- M. Pliev, M. Popov, *Representation theorems for regular operators*, Mathematische Nachrichten, DOI: 10.1002/mana.202200129.
- Н. М. Абасов, Н. А. Джусоева, М. А. Плиев, *Диффузные ортогонально аддитивные операторы*, Мат. сборник, **215** (2024), 1, 1–30.

We write $x = \bigsqcup_{i=1}^{n} x_i$ if $x = \sum_{i=1}^{n} x_i$ and $x_i \perp x_j$ for all $i \neq j$. In particular, if $n = 2$ we use the notation $x = x_1 \sqcup x_2$. We say that y is a *fragment* (a *component*) of $x \in E$, and use the notation $y \sqsubseteq x$, if $y \perp (x - y)$. The set of all fragments of an element $x \in E$ is denoted by $\mathcal{F}_x$. We say that $x_1, x_2 \in \mathcal{F}_x$ are *mutually complemented*, if $x = x_1 \sqcup x_2$.

Definition

Let E be a vector lattice and X a real vector space. A map $T\colon E \to X$ is called an *orthogonally additive operator* (OAO in short) provided $T(x + y) = T(x) + T(y)$ for any disjoint elements $x, y \in E$.

Definition

Let (A, Σ, μ) and (B, Ξ, ν) be finite measure spaces. By $(A \times B, \mu \otimes \nu)$ we denote the completion of their product measure space. The union $\Gamma \cup \Theta$ of two disjoint measurable sets $\Gamma, \Theta \in A$ we denote by $\Gamma \sqcup \Theta$. A map $K \colon A \times B \times \mathbb{R} \to \mathbb{R}$ is said to be a *Carathéodory function* if it satisfies the following conditions:

(C_1) $K(\cdot, \cdot, r)$ is $\mu \otimes \nu$-measurable for all $r \in \mathbb{R}$;

(C_2) $K(s, t, \cdot)$ is continuous on $\mathbb{R}$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

We say that a Carathéodory function K is *normalized* if $K(s, t, 0) = 0$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

Let E be an order ideal of $L_0(\nu)$, let $K\colon A \times B \times \mathbb{R} \to \mathbb{R}$ be a normalized Carathéodory function and for every $f \in E$ the function $K(s,\cdot,f(\cdot)) \in L_1(\nu)$ for almost all $s \in A$. Suppose that the function $s \mapsto \int_B K(s,t,f(t))\, d\nu(t)$ belongs to F. Then is defined orthogonally additive operator $T\colon E \to F$ by setting

$$Tf(s) = \int_B K(s,t,f(t))\, d\nu(t).$$

- M. A. Krasnosel'skii, P. P. Zabrejko, E. I. Pustil'nikov, P. E. Sobolevski, *Integral operators in spaces of summable functions*, Noordhoff, Leiden (1976)..

Let (A, Σ, μ) be a finite measure space. We say that $N : A \times \mathbb{R} \to \mathbb{R}$ is a *superpositionally measurable function*, or *sup-measurable* for brevity, if $N(\cdot, f(\cdot))$ is μ-measurable for every $f \in L_0(\mu)$. A sup-measurable function N is called *normalized* if $N(s, 0) = 0$ for μ-almost all $s \in A$.
With every normalized sup-measurable function N is associated an orthogonally additive operator $\mathcal{N} \colon L_0(\mu) \to L_0(\mu)$ defined by

$$\mathcal{N}(f)(s) = N(s, f(s)), \quad f \in L_0(\mu).$$

It is not hard to verify that $\mathcal{N}$ is a disjointness preserving operator. Indeed, for almost all $s \in A$ we have that

$$\mathcal{N}(f)(s) = N(s, f(s)) = N(s, f1_{\operatorname{supp} f}(s)) = N(s, f(s))1_{\operatorname{supp} f}(s).$$

Hence $\operatorname{supp} \mathcal{N}(f) \subset \operatorname{supp} f$ and $\mathcal{N}(f) \in \{f\}^{dd}$.
We note that the operator $\mathcal{N}$ is known in literature as the nonlinear superposition operator or Nemytskii operator.

- J. Appell, P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge, 1990.
- T. Runst, W. Sickel,*Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Series of Nonlinear Analysis and Applications, 1996.
- J. Appell, J. Banas, N. Merentes *Bounded variation and around*, De Gruyter, 2014.

A map $f\colon E \to \mathbb{R}$ is called valuation if

$$F(f \vee g) + F(f \wedge g) = F(f) + F(g), \quad f, g \in E$$

- M. Ludwig, M. Reitzner, *A classification of $SL(n)$ invariant valuation*, Annals Math., **172** (2010), 2, 1219–1267.
- S. Alesker, *Continuous rotation invariant valuations on convex sets*, Annals Math., **149** (1999), 3, 977–1005.
- P. Tradacete, and I. Villanueva, *Valuations on Banach lattices*,Int. Math. Res. Not., **2020** (2020), 1, 287–319.

Definition

Let E, F be vector lattices. An OAO $T : E \to F$ is said to be:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$;
- *regular* if $T = S_1 - S_2$, where S_1, S_2 are positive OAOs from E to F;
- *order bounded*, or an *abstract Uryson operator*, if it maps order bounded sets in E to order bounded sets in F;
- *disjointness preserving*, if $Tx \perp Ty$ for every disjoint $x, y \in E$;
- *C-bounded* or a *Popov operator*, if the set $T(\mathfrak{F}_x)$ is order bounded in F for every $x \in E$.

The sets of positive, regular, order bounded and C-bounded orthogonally additive operators from E to F we denote by $\mathcal{OA}_+(E,F)$, $\mathcal{OA}_r(E,F)$, $\mathcal{OA}_b(E,F)$ and $\mathcal{P}(E,F)$ respectively. There is a natural partial order on $\mathcal{OA}_r(E,F)$, namely $S \leq T \Leftrightarrow (T-S) \in \mathcal{OA}_+(E,F)$.

- The Riesz-Kantorovich type order calculus for regular OAOs were obtained. Note that for linear regular operators, this result is classical and fundamental. It was established by the founders of the theory of vector lattices F. Riesz and L.Kantorovich.

Theorem

Let E, F be vector lattices with F being Dedekind complete. Then $\mathcal{OA}_r(E,F)=\mathcal{P}(E,F)$, and $\mathcal{OA}_r(E,F)$ is a Dedekind complete vector lattice. Moreover, for every S, $T\in\mathcal{OA}_r(E,F)$ and every $x\in E$ the following hold:

1. $(T\vee S)x=\sup\{Ty+Sz:\ x=y\sqcup z\}$;
2. $(T\wedge S)x=\inf\{Ty+Sz:\ x=y\sqcup z\}$;
3. $T^{+}x=\sup\{Ty:\ y\sqsubseteq x\}$;
4. $T^{-}x=-\inf\{Ty:\ y\sqsubseteq x\}$;
5. $|Tx|\leq|T|x$.

- M. Pliev, K. Ramdane, *Order unbounded orthogonally additive operators in vector lattices*, Mediter. J. Math., **15** (2018), 2, article number 55.

Definition

Let E, F be vector lattices with F Dedekind complete. Regular OAO $T\colon E \to F$ is called *diffuse*, if $T \perp S$ for every $S \in \mathcal{OA}_{dpo}(E, F)$. The band of all diffuse OAOs we denote by $\mathcal{OA}_{dif}(E, F)$.

The band $\{\mathcal{OA}_{dpo}(E, F)\}^{\perp\perp}$ we denote by $\mathcal{OA}_{a}(E, F)$. The following decomposition take place

$$\mathcal{OA}_{r}(E, F) = \mathcal{OA}_{a}(E, F) \oplus \mathcal{OA}_{dif}(E, F).$$

Linear diffuse operators were studied in:

- *Е.В. Колесников.* // Диффузная и атомическая составляющие положительного оператора, Сиб. матем. журн., 39:2 (1998), 333-342; Siberian Math. J., 39:2 (1998), 292-300
- *C.B. Huijsmans, de Pagter.* // Disjointness preserving and diffuse operators, B. Compositio Mathematica, Tome 79 (1991) no. 3, pp. 351-374

It is worth to note that

$$\mathcal{OA}_{dif}(\mathbb{R}^n, \mathbb{R}^m) = \{0\}, \ m, n \in \mathbb{N}.$$

Nigel Kalton
1946-2010

Haskell Rosenthal
1940-2021

Kalton obtained the analytical representation of continuous operators on $L_1([0,1],\Sigma,\mu)$. This representation is given in the terms of weak*-measurable functions from $[0,1]$ to the set of all regular Borel measures on $[0,1]$.

- N. J. Kalton, The endomorphisms of L_p $(0 \le p \le 1)$, Indiana Univ. Math. J., **27** (1978), 3, 353–381.

Theorem

For every operator $T \in \mathcal{L}(L_1)$ *there is a weak** *measurable function* $\mu_s \colon [0,1] \to \mathcal{M}[0,1]$ *taking values in the space of all regular Borel measures on* $[0,1]$ *such that for every* $f \in L_1([0,1],\Sigma,\mu)$ *the following equality*

$$Tf(s) = \int_{[0,1]} f(t)d\,\mu_s(t) \tag{1}$$

holds for almost all $s \in [0,1]$*. Conversely, every weak**-*measurable function* $\mu \colon [0,1] \to \mathcal{M}[0,1]$ *defines an operator* $T \in \mathcal{L}(L_1)$ *as above.*

Supposing that $\mu_s^d = \sum\limits_{k=1}^{\infty} b_k(s)\delta_{\sigma_k(s)}$, where the sequence $(\sigma_k(s))_{k=1}^{\infty}$ is defined by

$$\mu_s^d([0,1]\setminus \bigcup_{k=1}^{\infty} \sigma_k(s)) = 0$$

we obtain that

$$Tf(s) = \int_{[0,1]} f(t)d\,\mu_s^a(t) + \sum_{k=1}^{\infty} b_k(s)f(\sigma_k(s)). \tag{2}$$

Hence, any operator $T \in \mathcal{L}(L_1)$ may be represented as the sum of two parts: "continuous" and "discrete" (atomic).

We say that a linear operator

$$T\colon L_1 \to L_1$$

1. *preserves disjointness* if for all $f, g \in L_1$ the relation

$$\mu\{t \in \operatorname{supp} f \cap \operatorname{supp} g\} = 0$$

implies that

$$\mu\{t \in \operatorname{supp} Tf \cap \operatorname{supp} Tg\} = 0;$$

2. *narrow* if for every $A \in \Sigma_+$ the restriction $T|_{L_1(A)}$ is not an isomorphic embedding to L_1;
3. *pseudo embedding* if for each $\varepsilon > 0$ there exists $A \in \Sigma_+$ such that the restriction $T|_{L_1(A)}$ is an into isomorphism.

Theorem

An operator $T \in \mathcal{L}(L_1)$ *is narrow if and only if for every* $D \in \Sigma_+$ *and* $\varepsilon > 0$ *there exists a decomposition* $D = D_1 \sqcup D_2$ *with disjoint* D_1 *and* D_2 *such that*

$$\|T(1_{D_1} - 1_{D_2}\| < \varepsilon.$$

We say that a linear operator

$$T \colon E \to F$$

between Banach lattices E and F is

1. *positive* if $T(E_+) \subset F_+$;
2. *regular* if $T = S_1 - S_2$ where S_1, S_2 are positive operators from E to F.

We state Rosenthal's representation theorem, as follows.

Theorem

Every operator $T \in \mathcal{L}(L_1)$ has a unique representation

$$T = T_{\mathcal{N}} + T_{\mathcal{H}},$$

where $T_{\mathcal{N}}$ is a narrow operator, $T_{\mathcal{H}}$ is a pseudo-embedding and $T_{\mathcal{N}}, T_{\mathcal{H}} \in \mathcal{L}(L_1)$.

Finally Maslyuchenko, Mykhaylyuk and Popov extended Theorem 8 to regular operators between Banach lattices.

O. Maslyuchenko, V. Mykhaylyuk, M. Popov, A lattice approach to narrow operators, Positivity, 13 (2009), 459–495.

They obtained the following remarkable result.

Theorem

Let E be an atomless Dedekind complete vector lattice, F an order ideal of some order continuous Banach lattice and $T : E \to F$ a regular order continuous operator. There then exists a unique decomposition

$$T = T_{\mathcal{N}} + T_{\mathcal{H}},$$

where $T_{\mathcal{N}}$ is an order continuous narrow operator and $T_{\mathcal{H}}$ is an order continuous atomic operator.

Definition

Let E be a vector lattice and Y be a normed space. An orthogonally additive operator $T\colon E \to Y$ is said to be *narrow* if for every $e \in E$ and $\varepsilon > 0$ there exist there exist a disjoint decomposition $e = e_1 \sqcup e_2$ such that $\|Tf_1 - Tf_2)\|_Y < \varepsilon$.

We say that $y_1, \dots, y_n$ is a *disjoint decomposition* of x, if $x = \bigsqcup_{i=1}^{n} y_i$. The set of all finite disjoint decompositions of x we denote by Θ_x. Let E, F be vector lattices with F Dedekind complete, $x \in E$ and $T \in \mathcal{OA}_r(E, F)$. We set

$$\mathfrak{p}_T(x) := \bigwedge_{\theta \in \Theta_x} \bigvee_{y \in \theta} |T|y, \quad x \in E.$$

Theorem

Let E, F be vector lattices with F Dedekind complete and $T : E \to F$ be a regular orthogonally additive operator. Then the following assertions are equivalent:

1. *T is a diffuse operator;*
2. *$\mathfrak{p}_T(x) = 0$ for every $x \in E$.*

- М. А. Плиев, М. М. Попов, *Продолжение абстрактных операторов Урысона*, Сиб. мат. жур., **57** (2016), 3, 700–708.
- V. Mykhaylyuk, M. Pliev, M. Popov, *The lateral order on Riesz spaces and orthogonally additive operators*, Positivity, **25** (2021), 2, 291-327.
- N. Erkursun-Özcan, M. Pliev, *On orthogonally additive operators in C-complete vector lattices*, Banach. J. Math. Anal., **16** (2022), 1, article number 6.

Theorem

Let E be a vector lattice and G an order continuous Banach lattice. Then an operator $T \in \mathcal{OA}_r(E, G)$ is narrow if and only if $|T|$ is.

Theorem

Let E be a vector lattice with the principal projection property and F be an order continuous Banach lattice. Then for every regular OAO $T : E \to F$ the following assertions are equivalent:

1. *T is diffuse;*
2. *T is narrow.*

Theorem

Let (A, Σ, μ), (B, Ξ, ν) *be finite measure spaces,* E, F *be order ideals in* $L_0(\nu)$ *and* $L_0(\mu)$ *respectively,* $T : E \to F$ *be an integral Uryson operator with the kernel* K*. Then the following statements are equivalent:*

1. T *is a regular OAO;*
2. *an orthogonally additive operator defined by the setting*

$$Rf(s) = \int_B |K(s, t, f(t))| d\nu(t), \;\; f \in E \tag{3}$$

acted from E *to* F*.*

Moreover for an operator $R = |T|$ *and every element* $f \in E$ *the following equalities hold:*

$$T^+ f = \int_B K(s, t, f(t))^+ d\nu(t); \;\; T^- f = \int_B K(s, t, f(t))^- d\nu(t). \tag{4}$$

Theorem

Пусть (A, Σ, μ) *be a finite measure space,* (B, Ξ, ν) *be a space with an atomless finite measure,* E, F *be order ideals in* $L_0(\nu)$ *and* $L_0(\mu)$ *respectively,* $T: E \to F$ *be an integral Uryson operator with the kernel* K*. Then* $T \in \mathcal{OA}_{dif}(E, F)$*.*

Theorem

Let E, F be vector lattices with F Dedekind complete, $x \in E$ and $T \in \mathcal{OA}_+(E,F)$. Then the order projection $\pi_a T$ in $\mathcal{OA}_r(E,F)$ onto the band of atomic operators can be calculated by the formula:

$$\pi_a T(x) = \sup\Big\{ \sum_{i=1}^{n} \mathfrak{p}_T(x_i) : \ x = \bigsqcup_{i=1}^{n} x_i\, ; \ n \in \mathbb{N} \Big\}. \tag{5}$$

THANK YOU FOR ATTENTION !